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Šapovalov elements for simple Lie algebras and basic classical simple Lie superalgebras.

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Abstract

Let $M(\lambda)$ be a Verma module for a basic classical simple Lie superalgebra $\mathfrak{g} \neq G(3)$ defined using the distinguished Borel subalgebra, and let γ be an isotropic positive root of $\mathfrak{g}$. As a special case of our first main result we show that if $\mu, \lambda \in \mathfrak{h}^*$ with $\lambda - \mu = \gamma$ we have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \leq 1.$$

This result applies to the construction of Šapovalov elements for isotropic roots. The proof rests on a comparison with the corresponding result for a certain simple Lie algebra G.

1 Introduction.

Throughout this paper we work over an algebraically closed field K of characteristic zero. We present a new connection between the representation theory of a basic classical simple Lie superalgebra $\mathfrak{g} \neq G(3)$ and a certain simple Lie algebra G. Specifically we are interested in Verma modules, highest weight vectors and Šapovalov elements for the two algebras.

The proofs of certain basic properties of Verma modules for a semisimple Lie algebra depend heavily on the fact that the enveloping algebra is a domain. However the enveloping algebra of a simple Lie superalgebra usually has zero divisors, and it is not clear that the proofs from the semisimple case can be adapted. For the study of Šapovalov elements it would be useful to know that any Verma module can contain, up to a scalar multiple, at most one highest weight vector with a fixed highest weight.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. As is well-known, there are in general several equivalence classes of bases of simple roots for $\mathfrak{g}$ under the action of the Weyl group. We choose a basis containing a unique isotropic root β. Verma modules are constructed using this basis of simple roots. Let Π_0 be the set of nonisotropic simple roots, $\overline{Q}_0^+ = \sum_{\alpha \in \Pi_0} \mathbb{N}\alpha$ and fix $\gamma \in \beta + \overline{Q}_0^+$. For example this will hold if γ is a

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positive isotropic root.

One of the main results of this paper is as follows. The remaining notation will be explained later.

Theorem A. *For any $\mu, \lambda \in \mathfrak{h}^*$ with $\lambda - \mu = \gamma$ we have*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \leq 1. \tag{1.1}$$

It was shown in [Mus12] Section 9.3 that (1.1) also holds provided $\lambda - \mu \in \overline{Q}_0^+$. Whether (1.1) holds in general is an open question, but we note that the analog of this inequality fails for parabolic Verma modules over simple Lie algebras, [IS88a].

Theorem A follows from another result, Theorem B, which provides a linear map ϕ from the space of vectors of weight $\lambda - \rho(\mathfrak{g}) - \gamma$ in $M(\lambda)$ to a suitable weight space in a Verma module over a simple Lie algebra G, such that ϕ preserves highest weight vectors. Roughly speaking the Dynkin diagram for G is obtained from the corresponding diagram for $\mathfrak{g}$ by replacing the unique grey node by a white node, but there are some exceptions to this rule for $\mathfrak{osp}(4,2)$. We delay the precise statement of Theorem B until subsection 2.8 since it requires the introduction of some notation first. In Theorem C we obtain a uniqueness and existence result for Šapovalov elements corresponding to isotropic roots of $\mathfrak{g}$. As an application of uniqueness we prove a result (Corollary 6.4) relating Šapovalov elements corresponding to isotropic roots to those corresponding to non-isotropic roots.

The enveloping algebras $U(\mathfrak{g})$ and $U(G)$ have markedly different properties in general. However we only need to consider the action of positive root vectors on weight spaces in a Verma module, with weights "close"' to the highest weight, and in this respect the two algebras are remarkably similar. We use the remainder of this introduction to provide some justification for this statement, and an outline of the proof, again delaying the introduction of the notation.

First observe that the (Dynkin-Kac) diagrams for the two algebras obtained by deleting the replaced nodes from the diagrams for $\mathfrak{g}, G$ are the same, and hence they correspond to isomorphic subalgebras $\mathfrak{p}$ and P of $\mathfrak{g}$ and G respectively. Next we use partitions to index a basis $\{e_{-\pi}\}$ of each weight space in $U(\mathfrak{n}^-)$. It turns out that the condition that π is a partition of γ, with γ as in Theorem A, places some strong conditions on π. For example if α is a non-isotropic root such that $\pi(\alpha) > 0$, then (in most cases, see Lemma 2.5 for the precise statement) α is a root of $\mathfrak{p}$. The next step is to deal with positive isotropic roots. We introduce a certain $\mathfrak{p}$-submodule $\mathfrak{c}$ of $\mathfrak{g}$ containing the root vectors for all such roots, and show that $\mathfrak{c}$ is isomorphic to a $\mathfrak{p}$-submodule C of G. After this has been done, it should be clear how to define the map ϕ in Theorem B on basis vectors, at least up to some scalar multiples, and then the rest of the proof consists of checking that it works.

2 Preliminaries.

2.1 Contragredient Lie superalgebras.

We first recall the construction of contragredient Lie superalgebras. Let $\mathbb{A} = (a_{ij})$ be an $n \times n$ matrix of rank l with entries in K. A *realization* of $\mathbb{A}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a vector space and

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \quad \Pi^\vee = \{h_1, \ldots, h_n\} \tag{2.1}$$

are linearly independent subsets of $\mathfrak{h}^*, \mathfrak{h}$ respectively, such that

$$\alpha_j(h_i) = a_{ij}. \tag{2.2}$$

By [Kac90], Proposition 1.1 the matrix $\mathbb{A}$ has a *minimal* realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, which is unique up to isomorphism. This condition means that $\dim \mathfrak{h} = 2n - \ell$. Now if τ is a subset of $\{1, 2, \ldots, n\}$, the Lie superalgebra $\widetilde{\mathfrak{g}}(\mathbb{A}, \tau)$ is generated by the vector space $\mathfrak{h}$ and elements e_i, f_i, h_i $(i = 1, \ldots, n)$ with $h_i \in \mathfrak{h}$ subject to the relations

$$[e_i, f_j] \;=\; \delta_{ij} h_i, \tag{2.3}$$

$$[h, h'] = 0, \tag{2.4}$$

$$[h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i \tag{2.5}$$

for $i, j = 1, \ldots, n$ and $h, h' \in \mathfrak{h}$. The $\mathbb{Z}_2$-grading on $\widetilde{\mathfrak{g}}(\mathbb{A}, \tau)$ is given by

$$\begin{aligned} \deg h_i = 0, \quad & \deg e_i = \deg f_i = 0 \quad \text{for } i \notin \tau \\ & \deg e_i = \deg f_i = 1 \quad \text{for } i \in \tau. \end{aligned}$$

There is a unique ideal $\mathfrak{r}$ of $\widetilde{\mathfrak{g}}$ which is maximal among all ideals of $\widetilde{\mathfrak{g}}$ which intersect $\mathfrak{h}$ trivially, and we set

$$\mathfrak{g} = \mathfrak{g}(\mathbb{A}, \tau) = \widetilde{\mathfrak{g}}(\mathbb{A}, \tau)/\mathfrak{r}. \tag{2.6}$$

Let $G(\mathbb{A}, \tau)$ be the derived subalgebra of $\mathfrak{g}$ and C the center of $G(\mathbb{A}, \tau)$. Let $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) be the subalgebra of $\mathfrak{g}$ generated by $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_n$). We say that two matrices A, A' indexed by the same set are *equivalent* if A' can be obtained by multiplying the rows of A by nonzero constants and relabeling the indices. Any basic classical simple Lie superalgebra can be constructed as a contragredient Lie superalgebra, usually using several different equivalence classes of matrices. However whenever a particular matrix $\mathbb{A}$ is specified we use the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+$ as above, and take the simple roots to be the α_i as in equation (2.1).

2.2 Cartan matrices and Lie superalgebras.

To describe our construction, it is easiest to begin with a simple Lie algebra G of rank $r > 1$ (or the Lie superalgebra $\mathfrak{osp}(1, 2r)$) with Cartan matrix $\mathbb{A}'$, and suppose that $1 \leq m < r$. We assume that if G is of type D then $m < r - 1$. Suppose that D is an invertible diagonal matrix such that $A' = D\mathbb{A}'$ is symmetric. It is easy to see the following.

Proposition 2.1. *There is a unique symmetric $r \times r$ matrix A such that*

(a) *The m^{th} diagonal entry of A equals 0,*

(b) *For $m < i \leq r$, the i^{th} row of A equals the i^{th} row of A',*

(c) *If $m > 1$, then for $i < m$ the i^{th} row of A equals -1 times the i^{th} row of A'.*

It follows from our construction that the set of simple roots of $\mathfrak{g}$ contains a unique isotropic root.

Example 2.2. Let $G = \mathfrak{sl}(2m)$, and let $A' = \mathbb{A}'$ be the Cartan matrix of G. If A is the matrix from Proposition 2.1, then $\mathfrak{g} = \mathfrak{g}(A, \{m\}) \cong \mathfrak{gl}(m, m)$. Note that A is the negative of the usual Cartan matrix for $\mathfrak{g}$, see for example [Mus12] Exercise 5.6.12. In this case we take the subalgebra $\mathfrak{h}$ of diagonal matrices as the Cartan subalgebra, and as generators e_i, f_i, h_i satisfying relations (2.3)- (2.5) we take the elements $e_i = e_{i,i+1}$ for $1 \leq i \leq 2m-1$,

$$h_i = \begin{cases} e_{i+1,i+1} - e_{i,i} & \text{if} \quad 1 \leq i \leq m-1 \\ -e_{m,m} - e_{m+1,m+1} & \text{if} \quad i = m \\ e_{i,i} - e_{i+1,i+1} & \text{if} \quad m+1 \leq i \leq 2m-1, \end{cases}$$

and

$$f_i = \begin{cases} -e_i^{\text{transpose}} & \text{if} \quad 1 \leq i \leq m \\ e_i^{\text{transpose}} & \text{if} \quad m+1 \leq i \leq 2m-1. \end{cases}$$

Let $(\ ,\ )$ be the invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$ defined by $(x, y) = -\text{Supertrace}\,(xy)$ for $x, y \in \mathfrak{g}$. In this case $\mathfrak{g}$ has a one-dimensional center C and $G(A, \{m\})/C \cong \mathfrak{psl}(m, m)$. Working with $\mathfrak{g} = \mathfrak{gl}(m, m)$ instead of $\mathfrak{psl}(m, m)$ in this case permits a more unified treatment, and as far as representation theory is concerned, there are only minor differences.

A natural question is, what are the Lie superalgebras that arise in this way? To answer this we describe our construction using Dynkin-Kac diagrams. Recall that the Dynkin-Kac diagram for a contragredient Lie superalgebra, is a graph with three kinds of nodes ○ , ⊗ and ● called *white, grey* and *black*, which correspond to even, isotropic and odd non-isotropic roots respectively. Nodes of the diagram correspond to simple roots, and are numbered from left to right. As explained in [Mus12] the Cartan matrix of a finite dimensional non-exceptional contragredient Lie superalgebra is determined by its square submatrices of size at most 3, and for this reason we give only the diagrams with at most 3 nodes. The information is summarized in the tables at the end of this paper. A few remarks are in order at this point. First, as mentioned before our construction also works if G is the simple Lie superalgebra $\mathfrak{osp}(1, 2r)$. In fact we adopt the following unconventional, but extremely useful rule. *From now on we use the term simple Lie algebra to mean either a Lie algebra from the usual A-G classification or the Lie superalgebra $\mathfrak{osp}(1, 2r)$ for some r.* Since the simple Lie algebras of type E_6, E_7, E_8 and G_2 have no Lie superalgebra analog which is finite dimensional contragredient under our construction, we assume that G is not isomorphic to any such algebra.

2.3 Invariant Bilinear Forms.

We assume that the matrix A is obtained from A' using Proposition 2.1, and set $\mathfrak{g} = \mathfrak{g}(A, \{m\})$. Since A' and $\mathbb{A}'$ are equivalent, $\mathfrak{g}(A', \emptyset) \cong \mathfrak{g}(\mathbb{A}', \emptyset)$. Also by [Car05] Corollary 14.16, $\mathfrak{g}(\mathbb{A}', \emptyset) \cong G$. An important role in this paper is played by the bilinear forms on Cartan subalgebras of G and $\mathfrak{g}$ defined by the matrices A' and A. It will be convenient to change our notation for the generators e_i, f_i for $\mathfrak{g}$ slightly. Thus if $\alpha = \alpha_i \in \Pi$ is a simple root of $\mathfrak{g}$, we shall write e_α and $e_{-\alpha}$ in place of e_i, f_i respectively. Note that equations (2.2), (2.3) and (2.5) imply that for $\alpha, \overline{\alpha} \in \Pi$,

$$[e_\alpha, e_{-\overline{\alpha}}] = \delta_{\alpha,\overline{\alpha}} h_\alpha, \quad [h_\alpha, e_{\overline{\alpha}}] = a_{\alpha,\overline{\alpha}} e_{\overline{\alpha}}, \quad [h_\alpha, e_{-\overline{\alpha}}] = -a_{\alpha,\overline{\alpha}} e_{-\overline{\alpha}}. \tag{2.7}$$

Let $\alpha_1, \dots, \alpha_r$ (resp. $\alpha'_1, \dots, \alpha'_r$) be the simple roots of $\mathfrak{g}$ (resp. G) corresponding to the nodes of the Dynkin-Kac diagrams, and define a linear map $\nu : \mathfrak{h}^* \longrightarrow H^*$ by $\nu(\alpha_i) = \alpha'_i$. Analogs for G of notions related to $\mathfrak{g}$ are generally denoted by the corresponding capital letter. In particular $N^\pm, H, B$ denote the analogs of $\mathfrak{n}^\pm, \mathfrak{h}, \mathfrak{b}$. The generators of G corresponding to $e_\alpha, e_{-\alpha}, h_\alpha$ are denoted $E_{\nu(\alpha)}, E_{-\nu(\alpha)}, H_{\nu(\alpha)}$. These elements satisfy the relations

$$[E_{\nu(\alpha)}, E_{-\nu(\overline{\alpha})}] = \delta_{\alpha,\overline{\alpha}} H_{\nu(\alpha)}, \quad [H_{\nu(\alpha)}, E_{\nu(\overline{\alpha})}] = a'_{\alpha,\overline{\alpha}} E_{\nu(\overline{\alpha})},$$

$$[H_{\nu(\alpha)}, E_{-\nu(\overline{\alpha})}] = -a'_{\alpha,\overline{\alpha}} E_{-\nu(\overline{\alpha})}. \tag{2.8}$$

We suppose next that $\mathfrak{g} \neq \mathfrak{gl}(m, m)$. By [Kac90] or [Mus12] Theorem 5.4.1, the Lie superalgebra $\mathfrak{g}$ has an invariant supersymmetric bilinear form $(\ ,\ )$ such that

$$(h_\alpha, h_{\overline{\alpha}}) = a_{\alpha\overline{\alpha}}, \ \ (e_\alpha, f_{\overline{\alpha}}) = \delta_{\alpha\overline{\alpha}}. \tag{2.9}$$

Furthermore the form $(\ ,\ )$ is nondegenerate and even, and the restriction of $(\ ,\ )$ to $\mathfrak{h}$ is nondegenerate. From (2.2) and (2.9) it follows that for any simple root α we have

$$(h_\alpha, h) = \alpha(h) \ \text{ for all } \ h \in \mathfrak{h}. \tag{2.10}$$

We use (2.10) to define elements $h_\alpha \in \mathfrak{h}$ for all $\alpha \in \mathfrak{h}^*$. Then the map $\alpha \longrightarrow h_\alpha$ gives a linear isomorphism from $\mathfrak{h}^*$ onto $\mathfrak{h}$.

We define nondegenerate symmetric bilinear forms $(\ ,\ )_\mathfrak{g}$, $(\ ,\ )_G$ on $\mathfrak{h}^*, H^*$ respectively by

$$(\alpha, \overline{\alpha})_\mathfrak{g} = a_{\alpha\overline{\alpha}}, \quad (\nu(\alpha), \nu(\overline{\alpha}))_G = a'_{\alpha\overline{\alpha}} \tag{2.11}$$

for all roots α, $\overline{\alpha}$.

Most of the modifications necessary to deal with the case where $\mathfrak{g} = \mathfrak{gl}(m, m)$ have been taken care of in Example 2.2. However since $\mathfrak{h}^*$ is not spanned by roots in this situation, we define elements $h_\alpha \in \mathfrak{h}$ by (2.10) corresponding to any $\alpha \in \mathfrak{h}^*$. Then the first equation in (2.11) continues to hold.

For α a non-isotropic simple root, we set $n_\alpha = \frac{(\alpha,\alpha)_\mathfrak{g}}{(\nu(\alpha),\nu(\alpha))_G}$, and $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)_\mathfrak{g}}$. Since the row of A indexed by α equals n_α times the corresponding row of A', it follows that

$$(\alpha, \overline{\alpha})_\mathfrak{g} = n_\alpha(\nu(\alpha), \nu(\overline{\alpha}))_G, \tag{2.12}$$

or equivalently

$$a_{\alpha,\overline{\alpha}} = n_\alpha a'_{\alpha,\overline{\alpha}}, \tag{2.13}$$

for any roots $\alpha, \overline{\alpha}$ of $\mathfrak{g}$ with α non-isotropic.

2.4 The exceptional Lie superalgebras $D(2,1;c)$.

Let A be the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -c \\ 0 & -c & 2c \end{bmatrix},$$

and set $D(2,1;c) = \mathfrak{g}(A, \{2\})$. In [Mus12] this algebra is called the exceptional Lie superalgebra $D(2,1;\alpha)$, where $\alpha = -c$, but we are using α to denote a root. If $c = -1, 2$ or $1/2$ then $\mathfrak{g} \cong D(2,1) = \mathfrak{osp}(4,2)$. Otherwise $\mathfrak{g}$ cannot be constructed using the previous method. However our main results still hold for $\mathfrak{g}$, and to adapt the proofs, we let A' be the Cartan matrix of $G = \mathfrak{sl}(4)$. We need the subalgebra $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is the subalgebra of $\mathfrak{g}$ generated by e_1, f_1 (resp. e_3, f_3). Thus $\mathfrak{g}_i \cong \mathfrak{sl}(2)$ for $i = 1, 2$. We define the map ν as before. Then (2.12) continues to hold, where now $n_\alpha = -1$ if α is a root of $\mathfrak{g}_1$, and $n_\alpha = c$ if α is a root of $\mathfrak{g}_2$.

2.5 Verma Modules.

Let $\Delta^+ = \Delta^+(\mathfrak{g})$ (resp. $\Delta^+(G)$) be the set of positive roots of $\mathfrak{g}$ and G respectively, and let $\rho_0(\mathfrak{g})$ (resp. $\rho_0(G)$) be the half sum of the positive even roots of $\mathfrak{g}$ (resp. of G). Similarly $\rho_1(\mathfrak{g})$ (resp. $\rho_1(G)$) denotes the half sum of the positive odd roots of $\mathfrak{g}$ (resp. of G). Then set $\rho(\mathfrak{g}) = \rho_0(\mathfrak{g}) - \rho_1(\mathfrak{g})$, and $\rho(G) = \rho_0(G) - \rho_1(G)$. We have triangular decompositions

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad G = N^- \oplus H \oplus N^+$$

of $\mathfrak{g}$ and of G respectively. Now for $\lambda \in \mathfrak{h}^*$, let $Kv_{\lambda-\rho(\mathfrak{g})}$ be the one dimensional $\mathfrak{b}$-module with trivial action of $\mathfrak{n}^+$ and weight $\lambda - \rho(\mathfrak{g})$, and define the Verma module $M(\lambda)$ by

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} Kv_{\lambda-\rho(\mathfrak{g})} \tag{2.14}$$

Similarly given $\Lambda \in H^*$, define the Verma module $M_G(\Lambda)$ for G with highest weight $\Lambda - \rho(G)$ by

$$M_G(\Lambda) = U(G) \otimes_{U(B)} Kv_{\Lambda-\rho(G)}.$$

2.6 Root Vectors.

Let $\mathfrak{p}$ and P be the semisimple subalgebras of $\mathfrak{g}$ and G generated by all root vectors $e_{\pm\alpha}$ and $E_{\pm\nu(\alpha)}$ respectively, where α is a non-isotropic simple root. The Dynkin diagram for $\mathfrak{p}$ is obtained from the Dynkin-Kac diagram for $\mathfrak{g}$ by deleting the m^{th} node. Removing the m^{th} node from the Dynkin diagram of G results in the same diagram. Thus $\mathfrak{p}$ and P are isomorphic Lie algebras, and we need to consider a particular isomorphism.

Lemma 2.3. *There is an isomorphism of Lie superalgebras $\theta : \mathfrak{p} \longrightarrow P$, such that for each non-isotropic simple root α,*

$$\theta(e_{-\alpha}) = E_{-\nu(\alpha)},\ \theta(e_\alpha) = n_\alpha E_{\nu(\alpha)} \text{ and } \theta(h_\alpha) = n_\alpha H_{\nu(\alpha)}. \tag{2.15}$$

Proof. This is a matter of checking that the defining relations are preserved, for example if $\alpha, \bar{\alpha}$ are both non-isotropic simple roots we have

$$[\theta(e_{\bar{\alpha}}), \theta(e_{-\alpha})] = \theta([e_{\bar{\alpha}}, e_{-\alpha}]) = \delta_{\alpha,\bar{\alpha}} n_\alpha H_{\nu(\alpha)},$$

and using (2.13),

$$[\theta(h_{\bar{\alpha}}), \theta(e_\alpha)] = \theta([h_{\bar{\alpha}}, e_\alpha]) = n_\alpha n_{\bar{\alpha}} a_{\alpha,\bar{\alpha}} E_{\nu(\alpha)}.$$

□

We extend θ to an isomorphism $U(\mathfrak{p}) \longrightarrow U(P)$ also denoted θ. This isomorphism is quite useful, because it means that certain computations to be carried out in $U(\mathfrak{p})$ will have counterparts in $U(P)$ that are easy to write down, see equations (2.16), (4.7) and (5.5). Moreover, it is due to the isomorphism θ that we are spared any case-by-case analysis.

Write $\mathfrak{p} = \mathbf{g}_1 \oplus \mathbf{g}_2$ where $\mathbf{g}_1 \cong \mathfrak{sl}(m)$, if $m > 1$ or $\mathbf{g}_1 = 0$ if $m = 1$, and $\mathbf{g}_2$ is a semisimple Lie algebra corresponding to the Dynkin diagram to the right of the deleted node. Now for each simple root α of $\mathfrak{p}$ and each positive integer t we have by (2.15), that for some polynomial p_t

$$[e_\alpha, e_{-\alpha}^t] = e_{-\alpha}^{t-1} p_t(h_\alpha), \text{ and } [E_{\nu(\alpha)}, E_{-\nu(\alpha)}^t] = n_\alpha^{-1} E_{-\nu(\alpha)}^{t-1} p_t(n_\alpha H_{\nu(\alpha)}). \tag{2.16}$$

For each non-simple root α of $\mathfrak{p}$ fix $e_{-\alpha} \in \mathfrak{p}^{-\alpha}$, and let $E_{-\nu(\alpha)} = \theta(e_{-\alpha}) \in P^{-\nu(\alpha)}$.

Let β be the unique isotropic simple root of $\mathfrak{g}$. Choose $e_{-\beta} \in \mathfrak{g}^{-\beta}$ and $E_{-\nu(\beta)} \in G^{-\nu(\beta)}$. Then let $\mathfrak{c}$ (resp. C) be the ad $\mathfrak{p}$-submodule of $\mathfrak{p}$ generated by $e_{-\beta}$, (resp. the ad P-submodule of G generated by $E_{-\nu(\beta)}$). We use the map θ to make C into a $\mathfrak{p}$-module.

Lemma 2.4. *As $\mathfrak{p}$-modules, $\mathfrak{c}$ and C are isomorphic.*

Proof. By equation (2.12) we have

$$\begin{aligned} \beta(h_\alpha) &= (\alpha,\beta)_{\mathfrak{g}} = n_\alpha(\nu(\alpha),\nu(\beta))_G \\ &= n_\alpha \nu(\beta)(H_{\nu(\alpha)}) = \nu(\beta)(\theta(h_\alpha)) \end{aligned} \tag{2.17}$$

for any non-isotropic root α of $\mathfrak{p}$ and hence $e_{-\beta} \in \mathfrak{c}$ and $E_{-\nu(\beta)} \in C$ are lowest weight vectors with the same weight. □

Next choose $e_{-\tau} \in \mathfrak{g}^{-\tau}$ and set $E_{-\nu(\tau)} = \theta(e_{-\tau}) \in G^{-\nu(\tau)}$. Since $\mathfrak{c}$ and C are generated as $\mathfrak{p}$-modules by repeated application of positive root vectors, we can extend θ to a map $\theta : \mathfrak{p} \oplus \mathfrak{c} \longrightarrow P \oplus C$ such that

$$\theta([e_{-\alpha}, e_{-\tau}]) = [E_{-\nu(\alpha)}, E_{-\nu(\tau)})], \tag{2.18}$$

for α a positive root of $\mathfrak{p}$, and $-\tau$ a root of $\mathfrak{c}$. Set $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{c}$ and $Q = P \oplus C$. Now $\mathfrak{q}$ need not be a subalgebra of $\mathfrak{g}$. Nevertheless we say that α is a root of $\mathfrak{q}$ (or $\mathfrak{c}$) if $\mathfrak{g}^\alpha \subset \mathfrak{q}$ or $\mathfrak{g}^\alpha \subset \mathfrak{c}$ respectively.

The roots are described explicitly in terms of certain linear functionals ϵ_i, δ_j in [Mus12]. The only case where Π contains an odd non-isotropic root arises when $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$, and the last node of the Dyknin-Kac diagram is black. In this case $\mathfrak{g}_0 \cong \mathfrak{o}(2m+1) \oplus \mathfrak{sp}(2n)$ and the short positive roots of $\mathfrak{o}(2m+1)$ are $\epsilon_1, \ldots, \epsilon_m$. Now we can describe the set of roots of $\mathfrak{c}$ explicitly.

Lemma 2.5.

(a) *If* Π *does not contain an odd non-isotropic root, then the roots of* $\mathfrak{c}$ *are precisely the odd positive roots of* $\mathfrak{g}$.

(b) *If* Π *contains an odd non-isotropic root, then the roots of* $\mathfrak{c}$ *are the odd positive roots of* $\mathfrak{g}$ *and the roots* $\epsilon_1, \ldots, \epsilon_m$.

Proof. Left to the reader. □

Set $Q^+ = \sum_{\alpha \in \Pi} \mathbb{N}\alpha$. If $\eta \in Q^+$, a *partition* of η is a map $\pi : \Delta^+ \longrightarrow \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for all $\alpha \in \Delta_1^+$ and

$$\sum_{\alpha \in \Delta^+} \pi(\alpha)\alpha = \eta.$$

If π is a partition of η, we set $|\pi| = \sum_{\alpha \in \Delta^+} \pi(\alpha)$ and $\|\pi\| = \eta$.

Lemma 2.6. *Let* π *be a partition of* γ*, and* α *a root such that* $\pi(\alpha) \neq 0$.

(a) α *is a root of* $\mathfrak{q}$.

(b) *If* α *is isotropic root then* $\pi(\alpha) = 1$.

Furthermore there is a unique isotropic root β' *such that* $\pi(\beta') > 0$ *and we have* $\pi(\beta') = 1$.

Proof. Suppose first that Π contains an odd non-isotropic root. Define a group homomorphism $\deg : \bigoplus \mathbb{Z}\epsilon_i \oplus \bigoplus \mathbb{Z}\delta_j \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ by setting $\deg \epsilon_i = (1,0), \deg \delta_j = (0,1)$ for all i,j. Then $\deg \gamma = (1,s)$ for some integer $s \geq -1$. The result follows from a consideration of the degrees of the positive roots. The other cases are similar. □

Corollary 2.7. *The dimensions of the vector spaces $U(\mathfrak{n}^-)^{-\gamma}$ and $U(N^-)^{-\nu(\gamma)}$ are equal.*

Proof. The dimension of $U(\mathfrak{n}^-)^{-\gamma}$ is the coefficient of $\varepsilon^{-\gamma}$ in

$$\frac{\prod_{\alpha \text{ a positive isotropic root of } \mathfrak{q}} (1+\varepsilon^{-\alpha})}{\prod_{\alpha \text{ a positive non-isotropic root of } \mathfrak{q}} (1-\varepsilon^{-\alpha})},$$

but this is the same as the coefficient of $\varepsilon^{-\nu(\gamma)}$ in

$$\prod_{\alpha \text{ a positive root of } Q} (1-\varepsilon^{-\alpha})^{-1},$$

and this equals $\dim_K U(N^-)^{-\nu(\gamma)}$. □

2.7 Weight Spaces in Verma Modules.

By Lemma 2.6 we now have a basis for all root spaces of $\mathfrak{g}$ and G that we will need. The next step is to introduce bases for the weight spaces in the Verma modules. As usual this is done using partitions. We denote by $\mathbf{P}(\eta)$ the set of partitions of η. We say the partition π is *non-isotropic* if $\pi(\alpha) = 0$ for all isotropic roots α. If σ is a partition of η for some $\eta \in (\gamma - Q^+) \cap Q^+$ we say that σ is a *sub-partition* of γ. Fix an ordering on the set Δ^+, of positive roots, and for π a sub-partition of γ, set

$$e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\alpha}^{\pi(\alpha)} \in U(\mathfrak{n}^-), \tag{2.19}$$

the product being taken with respect to this order. In addition set

$$E_{-\pi} = \prod_{\alpha \in \Delta^+} E_{-\nu(\alpha)}^{\pi(\alpha)} \in U(N^-), \tag{2.20}$$

where the product is taken in the same order. From now on we consider only partitions π such that $\pi(\beta') > 0$ for at most one isotropic root. In this case if $\pi(\beta') > 0$ for β' isotropic, we arrange that the root vectors $e_{-\beta'}$ and $E_{-\nu(\beta')}$ occur last in the products (2.19) and (2.20). If $\pi \in \mathbf{P}(\gamma)$ then we have a unique decomposition

$$e_{-\pi} = e_{-\zeta} e_{-\xi} e_{-\beta'}, \tag{2.21}$$

where ζ and ξ are sub-partitions of γ, $e_{-\zeta} \in U(\mathbf{g}_1), e_{-\xi} \in U(\mathbf{g}_2)$. Later, we fix an order on the factors of $e_{-\zeta}$, see subsection (3.1).

2.8 The map ϕ.

From now on fix $\lambda \in \mathfrak{h}^*$, and define $\Lambda \in H^*$ by requiring that $(\lambda, \alpha)_{\mathfrak{g}} = (\Lambda, \nu(\alpha))_G$ for every non-isotropic simple root α, and $(\lambda - \rho(\mathfrak{g}), \beta)_{\mathfrak{g}} = (\Lambda - \rho(G), \nu(\beta))_G$ For a sub-partition σ of γ define

$$r(\sigma) = \sum_{\alpha \text{ a root of } \mathbf{g}_1} \sigma(\alpha),$$

if $\mathfrak{g} \neq D(2,1;c)$. If $\mathfrak{g} = D(2,1;c)$ set

$$s_\sigma = \prod_{\alpha \text{ an even root of } \mathfrak{g},\ \sigma(\alpha) > 0} n_\alpha.$$

Theorem B. *There is a bijective linear map $\phi : M_{\mathfrak{g}}(\lambda)^{\lambda - \rho(\mathfrak{g}) - \gamma} \longrightarrow M_G(\Lambda)^{\Lambda - \rho(G) - \nu(\gamma)}$ which preserves highest weight vectors, given as follows*

$$\phi(e_{-\pi} v_{\lambda - \rho(\mathfrak{g})}) = (-1)^{r(\pi)} E_{-\pi} v_{\Lambda - \rho(G)} \quad \text{if } \mathfrak{g} \neq D(2,1;c), \tag{2.22}$$

$$\phi(e_{-\pi} v_{\lambda - \rho(\mathfrak{g})}) = s_\pi E_{-\pi} v_{\Lambda - \rho(G)} \quad \text{if } \mathfrak{g} = D(2,1;c).$$

From this point on we assume that $\mathfrak{g} \neq D(2,1;c)$ and prove Theorem B in this case only, leaving the reader to fill in the details when $\mathfrak{g} = D(2,1;c)$. All the definitions needed to do this have already been given, and in some respects the proof is easier for $\mathfrak{g} = D(2,1;c)$ due to the simpler structure of $\mathfrak{p}$ and $\mathfrak{c}$. We set $m(\alpha) = 1$ if α is a root of $\mathbf{g}_1$, and $m(\alpha) = 0$ if α is a root of $\mathbf{g}_2$. Then $n_\alpha = (-1)^{m(\alpha)}$.

Proof of Theorem B. The map is bijective by Corollary 2.7. Extending the above notation slightly, we now set $\phi(e_{-\sigma} v_{\lambda - \rho(\mathfrak{g})}) = (-1)^{r(\sigma)} E_{-\sigma} v_{\Lambda - \rho(G)}$, when σ is a sub-partition of γ. For the isotropic simple root β, set $n_\beta = 1$. Then it follows from equations (4.1) and (5.1) below that for any simple root α,

$$\phi\left(e_\alpha \cdot \sum_\pi a_\pi e_{-\pi} v_{\lambda - \rho(\mathfrak{g})}\right) = n_\alpha E_{\nu(\alpha)} \cdot \phi\left(\sum_\pi a_\pi e_{-\pi} v_{\lambda - \rho(\mathfrak{g})}\right). \tag{2.23}$$

□

The dot is inserted for clarity whenever a positive root vector acts on an element of a Verma module. Theorem A follows at once from the corresponding result for G and Theorem B.

Note that if $\mathfrak{g} \neq D(2,1;c)$, then (2.22) becomes

$$\phi(\sum_{\pi \in \mathbf{P}(\gamma)} a_\pi e_{-\pi} v_{\lambda - \rho(\mathfrak{g})}) \longrightarrow \sum_{\pi \in \mathbf{P}(\gamma)} (-1)^{r(\pi)} a_\pi E_{-\pi} v_{\Lambda - \rho(G)}. \tag{2.24}$$

3 Commutators.

3.1 The Lie algebra $\mathfrak{sl}(m)$.

In the proof of Theorem A we need to express certain commutators $[e_\alpha, e_{-\pi}]$ with α a simple root and $\pi \in \mathbf{P}(\gamma)$, as a linear combination of monomials as in (2.19). In doing this we need to compare $r(\eta)$ with $r(\pi)$ for each monomial $e_{-\eta}$ that arises. Now the definition of $r(\pi)$ depends only on roots from $\mathbf{g}_1$. So the main part of the computation takes place in $U(\mathbf{g}_1)$ where, since $\mathbf{g}_1 \cong \mathfrak{sl}(m)$, an explicit calculation can be made. Thus for the remainder of this subsection $\mathbf{g}_1$ will denote the Lie algebra $\mathfrak{sl}(m)$.

We say that a product p' of root vectors of $\mathbf{g}_1$ is equal to another such product p up to a *trivial rearrangement* of their factors, if p' can be obtained from p by exchanging the order of a sequence of pairs of commuting root vectors. We denote the usual matrix units in $\mathbf{g}_1$ by $e_{i,j}$. It turns out that with a suitable ordering of these matrix units, any commutators that arise are already multiples of monomials as in (2.19) up to a trivial rearrangement of their factors. We define an order on the $e_{i,j}$ with $i > j$ by saying that $e_{i,j} > e_{k,\ell}$ if either $i < k$ or $i = k$ and $j < \ell$.

For example, any monomial of the form $e_{2,1}^a e_{4,2}^b e_{4,3}^c e_{8,7}^d$ (with the usual condition on the exponents) has the form $e_{-\zeta}$ for a suitable partition ζ.

Lemma 3.1. *If α is a simple root of $\mathbf{g}_1$ and $\pi(\alpha) = 0$, then either $[e_\alpha, e_{-\pi}] = 0$, or $[e_\alpha, e_{-\pi}] = \sum_\eta d_\eta e_{-\eta}$ where the sum is over partitions η with $r(\eta) = r(\pi) - 1$. In this case we have $[E_{\nu(\alpha)}, E_{-\pi}] = \sum_\eta d_\eta E_{-\eta}$.*

Proof. By the Leibniz rule, it is enough to consider commutators involving the power $e_{i,j}^b$ of a root vector $e_{i,j}$ that occurs in $e_{-\pi}$. Under the stated hypotheses, $e_\alpha = e_{k,k+1}$ where $1 \le k < m$, and $[e_\alpha, e_{i,j}^b] \ne 0$ if and only if $k + 1 = i$, or $k = j$. Because $\pi(\alpha) = 0$, only one of these conditions can hold. In the first (resp. second) case we obtain a contribution to $[e_\alpha, e_{-\pi}]$ with $e_{i,j}^b$ replaced by $b e_{i-1,j} e_{i,j}^{b-1}$ (resp. by $-b e_{i,j}^{b-1} e_{i,j+1}$). Since all root vectors of the form $e_{i-1,p}$ (resp. $e_{i,p}$) commute, this proves the result after a trivial rearrangement of the factors. □

3.2 Commutators in $U(\mathfrak{g})$.

Suppose that α is a non-isotropic root, and that σ is a partition such that $\sigma(\alpha) > 0$. Define the partition $\overline{\sigma}$ by

$$\overline{\sigma}(\alpha') = \begin{cases} \sigma(\alpha) - 1 & \text{if} \quad \alpha' = \alpha \\ \sigma(\alpha') & \text{if} \quad \textit{otherwise.} \end{cases}$$

If $\sigma(\alpha) = 0$ we make the convention that $e_{-\overline{\sigma}} = 0$.

Set $S(\pi, \alpha) = \{\eta \in \mathbf{P}(\gamma - \alpha) : r(\eta) \equiv r(\pi) + m(\alpha) \mod 2\}$.

Lemma 3.2. *For a non-isotropic simple root α and partition σ we have*

$$[e_\alpha, e_{-\sigma}] = e_{-\overline{\sigma}}h + \sum_{\eta \in S(\sigma,\alpha):\eta \neq \overline{\sigma}} b_{\sigma,\eta} e_{-\eta}. \tag{3.1}$$

where the $b_{\sigma,\eta}$ are scalars, and h is a linear polynomial in h_α.

Proof. Since $[e_\alpha, e_{-\sigma}] \in U(\mathfrak{b}^-)$ it can be written as a linear combination of $e_{-\eta}$ with coefficients in $U(\mathfrak{h})$. Also, since α is a simple root, the only way a non-constant term from $U(\mathfrak{h})$ can arise is from a commutator of the form $[e_\alpha, e_{-\alpha}^{\sigma(\alpha)}]$ and this gives rise to the first term on the right of (3.1). For the remaining terms, note that if α is a root of $\mathbf{g}_1$, then by Lemma 3.1 $[e_\alpha, e_{-\sigma}]$ is after a trivial rearrangement a linear combination of $e_{-\eta}$ where $r(\eta) = r(\sigma) - 1$. Otherwise to rewrite $[e_\alpha, e_{-\sigma}]$ in the correct order gives rise only to commutators in $U(\mathbf{g}_2)$, so we need only use partitions η with $r(\eta) = r(\sigma)$. □

4 The non-isotropic case.

The goal of this section is to show

Proposition 4.1. *If α is a non-isotropic simple root and $\pi \in \mathbf{P}(\gamma)$, then*

$$\phi(e_\alpha \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) = n_\alpha E_{\nu(\alpha)} \cdot \phi(e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}). \tag{4.1}$$

Write

$$e_{-\pi} = e_{-\sigma} e_{-\alpha}^t e_{-\tau} \tag{4.2}$$

where σ and τ are partitions satisfying $\sigma(\alpha) = \tau(\alpha) = 0$. Then

$$e_\alpha \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})} = k_\pi v_{\lambda-\rho(\mathfrak{g})} + \ell_\pi v_{\lambda-\rho(\mathfrak{g})}, \tag{4.3}$$

where, using (2.16)

$$k_\pi = ([e_\alpha, e_{-\sigma}] e_{-\alpha}^t e_{-\tau} + e_{-\sigma} e_{-\alpha}^t [e_\alpha, e_{-\tau}]), \quad \ell_\pi = e_{-\sigma} e_{-\alpha}^{t-1} p_t(h_\alpha) e_{-\tau}.$$

Note that $e_{-\sigma} e_{-\alpha}^{t-1} e_{-\tau} = e_{-\zeta}$ where ζ is a sub-partition of γ such that

$$r(\zeta) + m_\alpha = r(\pi). \tag{4.4}$$

On the other hand we have,

$$\begin{aligned} E_{\nu(\alpha)} \cdot \phi(e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) &= (-1)^{r(\pi)} E_{\nu(\alpha)} \cdot E_{-\pi} v_{\Lambda-\rho(G)} \\ &= (-1)^{r(\pi)} (K_\pi + L_\pi) v_{\Lambda-\rho(G)}, \end{aligned} \tag{4.5}$$

where

$$K_\pi = ([E_{\nu(\alpha)}, E_{-\nu(\sigma)}] E_{-\nu(\alpha)}^t E_{-\nu(\tau)} + E_{-\nu(\sigma)} E_{-\nu(\alpha)}^t [E_{\nu(\alpha)}, E_{-\nu(\tau)}]),$$

and by (2.16)

$$L_\pi = E_{-\nu(\sigma)} E_{-\nu(\alpha)}^{t-1} p_t(H_\alpha) E_{-\nu(\tau)}.$$

Now by Lemma 3.2 we have

$$k_\pi = \sum_{\eta \in S(\pi,\alpha)} b_{\pi,\eta} e_{-\eta}, \tag{4.6}$$

and since $\theta : \mathfrak{p} \longrightarrow P$ is an isomorphism of Lie superalgebras, satisfying (2.15) it follows that

$$K_\pi = n_\alpha \sum_{\eta \in S(\pi,\alpha)} b_{\pi,\eta} E_{-\nu(\eta)}. \tag{4.7}$$

The factor n_α in (4.7) comes from the presence of the term e_α in the definition of k_π.

Now noting that π determines τ and t in (4.2) set

$$\omega_\pi = p_t((\alpha, \lambda - \rho(\mathfrak{g}) - \|\tau\|)_\mathfrak{g}) = p_t(n_\alpha(\nu(\alpha), \Lambda - \rho(G) - \nu(\|\tau\|))_G).$$

Then

$$\ell_\pi v_{\lambda-\rho(\mathfrak{g})} = \omega_\pi e_{-\sigma} e_{-\alpha}^{t-1} e_{-\tau} v_{\lambda-\rho(\mathfrak{g})}, \tag{4.8}$$

and

$$L_\pi v_{\Lambda-\rho(G)} = n_\alpha \omega_\pi E_{-\sigma} E_{-\alpha}^{t-1} E_{-\tau} v_{\Lambda-\rho(G)}. \tag{4.9}$$

Combining equation (4.8) with equations (4.3) and (4.6) we have

$$e_\alpha \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})} = \omega_\pi e_{-\sigma} e_{-\alpha}^{t-1} e_{-\tau} v_{\lambda-\rho(\mathfrak{g})} + \sum_{\eta \in S(\pi,\alpha)} b_{\pi,\eta} e_{-\eta} v_{\lambda-\rho(\mathfrak{g})}. \tag{4.10}$$

Applying ϕ and noting (4.4) we have

$$\begin{aligned} \phi(e_\alpha \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) &= (-1)^{r(\pi)+m(\alpha)} \omega_\pi E_{-\nu(\sigma)} E_{-\nu(\alpha)}^{t-1} E_{-\nu(\tau)} v_{\Lambda-\rho(G)} \\ &+ \sum_{\eta \in S(\pi,\alpha)} (-1)^{r(\pi)+m(\alpha)} b_{\pi,\eta} E_{-\nu(\eta)} v_{\Lambda-\rho(G)}. \end{aligned} \tag{4.11}$$

On the other hand combining equation (4.9) with equation (4.7) as in the proof of equation (4.10) we have

$$E_{\nu(\alpha)} \cdot E_{-\pi} v_{\Lambda-\rho(G)} = n_\alpha \omega_\pi E_{-\nu(\sigma)} E_{-\nu(\alpha)}^{t-1} E_{-\nu(\tau)} v_{\Lambda-\rho(G)} + n_\alpha \sum_{\eta \in S(\pi,\alpha)} b_{\pi,\eta} E_{-\nu(\eta)} v_{\Lambda-\rho(G)}. \tag{4.12}$$

Comparing equations (4.11) and (4.12), and taking note of (4.5), we see that equation (4.1) holds in this case.

5 The isotropic case.

In this section we prove the following counterpart to Proposition 4.1 for the isotropic simple root.

Proposition 5.1. *If β is the unique simple isotropic root, and $\pi \in \mathbf{P}(\gamma)$, then*

$$\phi(e_\beta \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) = E_{\nu(\beta)} \cdot \phi(e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}). \tag{5.1}$$

We can write $e_{-\pi} = e_{-\hat{\pi}}e_{-\beta'}$ and $E_{-\pi} = E_{-\hat{\pi}}E_{-\nu(\beta')}$ where $\hat{\pi}$ is an non-isotropic partition, and β' is the unique odd root of $\mathfrak{c}$ such that $\pi(\beta') > 0$. Thus $[e_\beta, e_{-\hat{\pi}}] = [E_{\nu(\beta)}, E_{-\hat{\pi}}] = 0$, and $[e_\beta, e_{-\beta'}] = [E_{\nu(\beta)}, E_{-\nu(\beta')}] = 0$ unless $\beta' = \beta$ or $\beta' - \beta$ is a root. We consider each case separately.

If $\beta' = \beta$ then

$$e_\beta \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})} = e_{-\hat{\pi}} h_\beta v_{\lambda-\rho(\mathfrak{g})} = (\beta, \lambda - \rho(\mathfrak{g}))_{\mathfrak{g}} e_{-\hat{\pi}} v_{\lambda-\rho(\mathfrak{g})},$$

and so

$$\phi(e_\beta \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) = (-1)^{r(\pi)}(\beta, \lambda - \rho(\mathfrak{g}))_{\mathfrak{g}} E_{-\hat{\pi}} v_{\Lambda-\rho(G)}.$$

On the other hand

$$\begin{aligned} E_{\nu(\beta)} \cdot \phi(e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) &= (-1)^{r(\pi)} E_{\nu(\beta)} \cdot E_{-\pi} v_{\Lambda-\rho(G)} \\ &= (-1)^{r(\pi)}(\nu(\beta), \Lambda - \rho(G))_G E_{-\hat{\pi}} v_{\Lambda-\rho(G)}. \end{aligned}$$

Comparing the last two equations and using the definition of Λ gives equation (5.1) in this case.

Now suppose that $\beta' - \beta = \alpha$ is a non-isotropic root. Then by equation (2.18) $e_{-\beta-\alpha} = c_\alpha[e_{-\alpha}, e_{-\beta}]$ and $E_{-\nu(\beta+\alpha)} = c_\alpha[E_{-\nu(\alpha)}, E_{-\nu(\beta)}]$ for some non-zero scalar c_α.

Lemma 5.2. *With the above notation, we have*

$$e_\beta \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})} = n_\alpha(\nu(\beta), \nu(\alpha))_G c_\alpha e_{-\hat{\pi}} e_{-\alpha} v_{\lambda-\rho(\mathfrak{g})}, \tag{5.2}$$

and

$$E_{\nu(\beta)} \cdot E_{-\pi} v_{\Lambda-\rho(G)} = (\nu(\beta), \nu(\alpha))_G c_\alpha E_{-\hat{\pi}} E_{-\nu(\alpha)} v_{\Lambda-\rho(G)}. \tag{5.3}$$

Proof. The Jacobi identity gives

$$\begin{aligned} [e_\beta, e_{-\beta-\alpha}] &= c_\alpha[e_\beta, [e_{-\alpha}, e_{-\beta}]] = -c_\alpha[h_\beta, e_{-\alpha}] \\ &= c_\alpha(\beta, \alpha)_{\mathfrak{g}} e_{-\alpha} = n_\alpha(\nu(\beta), \nu(\alpha))_G c_\alpha e_{-\alpha}. \end{aligned}$$

Since $e_\beta \cdot v_{\lambda-\rho(\mathfrak{g})} = 0$, this gives (5.2), and the proof of (5.3) is similar. □

Now if α is a root of $\mathbf{g}_1$, then since $\beta + \alpha$ is a root, we have $e_{-\alpha} = e_{m,a}$ for some a. Since all root vectors of the form $e_{m,p}$ commute, the proof of Lemma 3.2 shows that in $U(\mathfrak{n}^-)$,

$$e_{-\hat{\pi}} e_{-\alpha} = \sum_{\eta : r(\eta) = r(\pi) + m_\alpha} b'_{\pi,\eta} e_{-\eta}, \tag{5.4}$$

and so applying θ

$$E_{-\hat{\pi}} E_{-\nu(\alpha)} = \sum_{\eta : r(\eta) = r(\pi) + m_\alpha} b'_{\pi,\eta} E_{-\eta}. \tag{5.5}$$

From (5.2) and (5.4) we have,

$$\begin{aligned}\phi(e_\beta \cdot e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) &= n_\alpha(\nu(\beta),\nu(\alpha))_G c_\alpha \sum_{\eta: r(\eta)=r(\pi)+m_\alpha} (-1)^{r(\eta)} b'_{\pi,\eta} E_{-\eta} v_{\lambda-\rho(\mathfrak{g})} \\ &= (-1)^{r(\pi)}(\nu(\beta),\nu(\alpha))_G c_\alpha \sum_{\eta: r(\eta)=r(\pi)+m_\alpha} b'_{\pi,\eta} E_{-\eta} v_{\lambda-\rho(\mathfrak{g})} \quad (5.6)\end{aligned}$$

Similarly using (5.3) and (5.5),

$$\begin{aligned}E_{\nu(\beta)} \cdot \phi(e_{-\pi} v_{\lambda-\rho(\mathfrak{g})}) &= (-1)^{r(\pi)} E_{\nu(\beta)} \cdot E_{-\pi} v_{\Lambda-\rho(G)} \quad (5.7) \\ &= (-1)^{r(\pi)}(\nu(\beta),\nu(\alpha))_G c_\alpha \sum_{\eta: r(\eta)=r(\pi)+m_\alpha} b'_{\pi,\eta} E_{-\nu(\eta)} v_{\Lambda-\rho(G)}.\end{aligned}$$

Comparing equation (5.6) to (5.7), we obtain equation (5.1).

6 Šapovalov elements.

Let G be a semisimple Lie algebra or the Lie superalgebra $\mathfrak{osp}(1,2r)$. To state the next result it is helpful to introduce a modified notation for partitions. For $\eta \in Q^+$ and define $\overline{\mathbf{P}}(\eta)$ to be the set of all maps $\pi : \Delta^+ \longrightarrow \mathbb{N}$ such that (2.19) holds, and in addition $\pi(\alpha) = 0$ or 1 for all positive isotropic roots α, and $\pi(\alpha) = 0$ whenever α is a root such that $\alpha/2$ is an odd root. Now suppose the set Δ^+ is ordered in such a way that 2α follows α whenever α is an odd non-isotropic root. Let f be the bijection from $\mathbf{P}(\eta)$ to $\overline{\mathbf{P}}(\eta)$ defined in [Mus12] Remark 8.4.3. If $\sigma = f(\pi)$ we define $e_{-\sigma}$ as in Equations (2.19) and (2.20) with π replaced by σ. It is clear that $e_{-\sigma}$ is a nonzero constant multiple of $e_{-\pi}$.

Let γ be a non-isotropic root, and let $\pi^0 \in \overline{\mathbf{P}}(p\gamma)$ be the unique partition of $p\gamma$ such that $\pi^0(\alpha) = 0$ for each non-simple root α. Let $(\Lambda, \gamma^\vee) = p$. Suppose that one of the following holds

$$p \text{ is an odd positive integer if } \gamma \text{ is an odd root,} \quad (6.1)$$

$$p \text{ is a positive integer if } \gamma \text{ is an even root such that } \gamma/2 \text{ is not a root.} \quad (6.2)$$

For an element u in the Weyl group W, let $\ell(u)$ be the length of u and set $N(u) = \{\alpha \in \Delta_0^+ | u\alpha < 0\}$. Suppose that $\gamma = w\beta$ for some $w \in W$ and simple root β. For $\alpha \in N(w^{-1})$, let $q(w,\alpha) = (w\beta, \alpha^\vee)$. For α a positive root, and then for π a partition, we define as in [Mus] the *Clifford degree* of α, π by

$$\mathrm{Cdeg}(\alpha) = 2 - i, \text{ for } \alpha \in \Delta_i^+, \quad \mathrm{Cdeg}(\pi) = \sum_{\alpha \in \Delta^+} \pi(\alpha)\mathrm{Cdeg}(\alpha).$$

For G a simple Lie algebra, parts (a) and (b) of the result below are due to Šapovalov, [Šap72].

Theorem 6.1.

(a) *The weight space* $M_G(\Lambda)^{\Lambda-\rho(G)-p\gamma}$ *contains a highest weight vector* $\theta^\Lambda_{\gamma,p}v_{\Lambda-\rho(G)}$ *which is unique up to a scalar multiple.*

(b) $\theta^\Lambda_{\gamma,p} = \sum_{\pi\in\overline{\mathbf{P}}(p\gamma)} a_\pi(\Lambda)e_{-\pi}v_\Lambda$ *where* $a_\pi(\Lambda) \in K$ *depends polynomially on* Λ, *and* $a_{\pi^0} = 1$.

(c) *If* G *is a simple Lie algebra, then* $a_{p\pi_\gamma}(\Lambda)$ *has leading term* $\prod_{\alpha\in N(w^{-1})}\Lambda(h_\alpha)^{pq(w,\alpha)}$, *and for all* $\pi \in \overline{\mathbf{P}}(p\gamma)$, $|\pi| + \deg a_\pi(\Lambda) \le p\mathrm{ht}\gamma$.

(d) *If* $G = \mathfrak{osp}(1,2r)$, *and* $p = 1$, $a_{\pi_\gamma}(\Lambda)$ *has leading term* $\prod_{\alpha\in N(w^{-1})}\Lambda(h_\alpha)$, *and for all* $\pi \in \overline{\mathbf{P}}(\gamma)$, $\mathrm{Cdeg}(\pi) + 2\deg a_\pi(\Lambda) \le 2\ell(w) + 1$.

Proof. The existence of and the polynomial nature of the coefficients follows from [Mus12] Theorem 9.2.6. Uniqueness follows from [Mus12] Theorem 9.3.1 or 9.3.6. Statements (c) and (d) are taken from [Mus]. □

Lemma 6.2. *We have* $(\lambda,\gamma)_\mathfrak{g} = 0$ *if and only if* $(\Lambda, \nu(\gamma)^\vee)_G = 1$.

Proof. Write $\gamma = \beta + \sum a_\alpha\alpha$ where the sum is over the non-isotropic simple roots α of $\mathfrak{g}$. An easy calculation shows that $(\Lambda,\nu(\gamma))_G = (\lambda,\gamma)_\mathfrak{g} + (\rho(G),\nu(\beta))$, so the result follows since $\nu(\beta)$ and $\nu(\gamma)$ have the same length. □

Now suppose $\mathfrak{g} = \mathfrak{g}(A,\{m\})$ as in subsection 2.3, and that $\mathfrak{g} \ne G(3)$. The Verma module $M(\lambda)$ is defined as in (2.14).

Theorem C. Let γ be a positive isotropic root of $\mathfrak{g}$, and suppose $(\lambda,\gamma)_\mathfrak{g} = 0$. Then

(a) The weight space $M(\lambda)^{\lambda-\rho(\mathfrak{g})-\gamma}$ contains a highest weight vector $\theta^\lambda_\gamma v_{\lambda-\rho(\mathfrak{g})}$ which is unique up to a scalar multiple.

(b) $\theta^\lambda_\gamma = \sum_{\pi\in\overline{\mathbf{P}}(\gamma)} a_\pi(\lambda)e_{-\pi}v_\lambda$ where $a_\pi(\lambda) \in K$ depends polynomially on λ, and $a_{\pi^0} = 1$.

(c) If Π contains no odd non-isotropic root, then $|\pi| + \deg a_\pi(\lambda) \le \mathrm{ht}\gamma$ for all $\pi \in \overline{\mathbf{P}}(\gamma)$, and $a_{\pi_\gamma}(\lambda)$ has leading term $\prod_{\alpha\in N(w^{-1})}\lambda(h_\alpha)^{q(w,\alpha)}$.

(d) If Π contains an odd non-isotropic root, then $\mathrm{Cdeg}(\pi)+2\deg a_\pi(\lambda) \le 2\ell(w)+1$, for all $\pi \in \overline{\mathbf{P}}(\gamma)$, and $a_{\pi_\gamma}(\lambda)$ has leading term $\prod_{\alpha\in N(w^{-1})}\lambda(h_\alpha)$.

Proof. This follows at once from Theorem 6.1 with $p = 1$, Lemma 6.2 and Theorem B. □

The next result follows immediately from the definitions.

Corollary 6.3. *In Theorem C, if* $\pi \ne \pi_\gamma$ *then* $a_\pi(\lambda) < \deg a_{\pi_\gamma}(\lambda)$.

The above results can be reformulated using the universal Verma module as in [Mus12]. First we define hyperplanes in $\mathfrak{h}^*$ as follows. For each non-isotropic root α, and integer p set

$$\mathcal{H}_{\alpha,p} = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha) = p(\alpha, \alpha)/2\}.$$

Similarly for an odd isotropic root γ we let

$$\mathcal{H}_{\gamma} = \{\lambda \in \mathfrak{h}^* | (\lambda, \gamma) = 0\}.$$

Now suppose that $\mathcal{H} = \mathcal{H}_{\alpha,p}$ or $\mathcal{H}_{\gamma}$. Let $\mathcal{I}(\mathcal{H})$ be the defining ideal of $\mathcal{H}$, and $\mathcal{O}(\mathcal{H}) = S(\mathfrak{h})/\mathcal{I}(\mathcal{H})$ the ring of polynomial functions on $\mathcal{H}$. Then if $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$

$$M = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}^+ = U(\mathfrak{b}^-) = U(\mathfrak{n}^-) \otimes S(\mathfrak{h})$$

is the universal Verma module. Given $\lambda \in \mathfrak{h}^*$ we define the *specialization* at λ to be the map

$$\varepsilon^{\lambda} : M \longrightarrow M(\lambda), \quad \sum_i a_i \otimes b_i \longrightarrow \sum_i a_i b_i(\lambda) v_{\lambda - \rho(\mathfrak{g})}.$$

By lifting the polynomials $a_\pi(\lambda)$ to $H_\pi \in S(\mathfrak{h})$ such that $H_\pi(\lambda) = a_\pi(\lambda)$, we obtain $\theta_{\alpha,p} = \sum_{\pi \in \overline{\mathbf{P}}(p\alpha)} e_{-\pi} H_\pi$ and $\theta_\gamma = \sum_{\pi \in \overline{\mathbf{P}}(\gamma)} e_{-\pi} H_\pi$ of M such that $\theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})}$ and $\theta_\gamma v_{\lambda-\rho(\mathfrak{g})}$ are highest weight vectors

These elements $\theta_{\alpha,p}$ and θ_γ are unique modulo the left ideal $U(\mathfrak{b}^-)\mathcal{I}(\mathcal{H})$ of $U(\mathfrak{b}^-)$. In the next result, we set $\theta_{\alpha,0} = 1$.

Corollary 6.4. *Let γ be a positive isotropic root and α a non-isotropic root contained in $\overline{Q}_0^+$. Let $v_{\lambda-\rho(\mathfrak{g})}$ be a highest weight vector in a Verma module with highest weight λ, and set $\gamma' = s_\alpha \gamma$. Let $p = (\lambda, \alpha^\vee)$, and suppose $q = (\gamma, \alpha^\vee) \in \mathbb{N}\backslash\{0\}$. Assume that* (6.1) or (6.2) holds when γ is replaced by α. Then

(a) If $(\lambda, \gamma')_{\mathfrak{g}} = 0$ we have

$$\theta_\gamma \theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})} = \theta_{\alpha,p+q} \theta_{\gamma'} v_{\lambda-\rho(\mathfrak{g})}.$$

(b) If $(\lambda, \gamma)_{\mathfrak{g}} = 0$, and $p - q \geq 0$, we have

$$\theta_{\gamma'} \theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})} = \theta_{\alpha,p-q} \theta_{\gamma} v_{\lambda-\rho(\mathfrak{g})}.$$

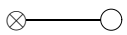
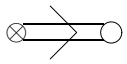
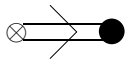
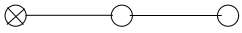
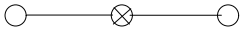
Proof. Under the hypothesis in (a) $\theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})}$ is a highest weight vector of weight $s_\alpha \cdot \lambda$ and $(s_\alpha \cdot \lambda + \rho, \gamma) = 0$. Therefore $\theta_\gamma \theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})}$ is a highest weight vector of weight

$$\lambda - \eta = s_\alpha \cdot \lambda - \gamma = s_\alpha \cdot (\lambda - \gamma'),$$

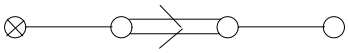
but $\theta_{\alpha,p+q} \theta_{\gamma'} v_{\lambda-\rho(\mathfrak{g})}$ is a highest weight vector with the same weight. Hence by Theorem C (a), $\theta_\gamma \theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})}$ and $\theta_{\alpha,p+q} \theta_{\gamma'} v_{\lambda-\rho(\mathfrak{g})}$ are equal up to a scalar multiple. Now if π^0 is the partition of η with $\pi^0(\sigma) = 0$ for all non-simple roots σ, then it follows easily from Theorem C (b), that $e_{-\pi^0} v_{\lambda-\rho(\mathfrak{g})}$ occurs with coefficient equal to one in both $\theta_\gamma \theta_{\alpha,p} v_{\lambda-\rho(\mathfrak{g})}$ and $\theta_{\alpha,p+q} \theta_{\gamma'} v_{\lambda-\rho(\mathfrak{g})}$, and from this we obtain the desired conclusion. The proof of (b) is similar. □

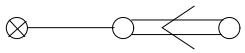
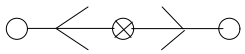
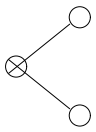
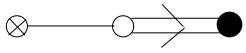
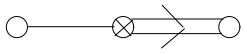
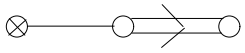
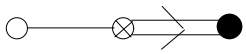
Remark 6.5. The proof shows that the conclusion in (a) still holds if α is an even root such that $\alpha/2$ is not a root, and $p = 0$.

7 Tables for Low Ranks.

$[m, G, \mathfrak{g}]^{\text{transpose}}$	Cartan matrix for G	Cartan matrix for $\mathfrak{g}$	Diagram for $\mathfrak{g}$
$\begin{bmatrix} 1 \\ \mathfrak{sl}(3) \\ \mathfrak{sl}(2,1) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{o}(5) \\ \mathfrak{osp}(3,2) \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{osp}(1,4) \\ \mathfrak{osp}(3,2) \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{sl}(4) \\ \mathfrak{sl}(1,3) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ \mathfrak{sl}(4) \\ \mathfrak{gl}(2,2) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	

$[m, G, \mathfrak{g}]^{\text{transpose}}$	Cartan matrix for G
$\begin{bmatrix} 1 \\ F_4 \\ F(4) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

Cartan matrix for $\mathfrak{g}$	Diagram for $\mathfrak{g}$
$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$	

$[m, G, \mathfrak{g}]^{\text{transpose}}$	Cartan matrix for G	Cartan matrix for $\mathfrak{g}$	Diagram for $\mathfrak{g}$
$\begin{bmatrix} 1 \\ \mathfrak{o}(7) \\ \mathfrak{osp}(2,4) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ \mathfrak{o}(7) \\ \mathfrak{osp}(4,2) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 4 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{o}(6) \\ \mathfrak{osp}(4,2) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{osp}(1,6) \\ \mathfrak{osp}(3,4) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ \mathfrak{sp}(6) \\ \mathfrak{osp}(3,4) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	
$\begin{bmatrix} 1 \\ \mathfrak{sp}(6) \\ \mathfrak{osp}(5,2) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	
$\begin{bmatrix} 2 \\ \mathfrak{osp}(1,6) \\ \mathfrak{osp}(5,2) \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	

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